Chapter 7  

Weighted Averages

This chapter addresses the problem of combining two or more separate and independent measurements of a single physical quantity. We will find that the best estimate of that quantity, based on the several measurements, is an appropriate weighted average of those measurements.

7.1 The Problem of Combining Separate Measurements

Often, a physical quantity is measured several times, perhaps in several separate laboratories, and the question arises how these measurements can be combined to give a single best estimate. Suppose, for example, that two students, A and B, measure a quantity $x$ carefully and obtain these results:

\[ x = x_A \pm \sigma_A \quad (7.1) \]

and

\[ x = x_B \pm \sigma_B. \quad (7.2) \]

Each result will probably itself be the result of several measurements, in which case $x_A$ will be the mean of all A's measurements and $\sigma_A$ the standard deviation of that mean (and similarly for $x_B$ and $\sigma_B$). The question is how best to combine $x_A$ and $x_B$ for a single best estimate of $x$.

Before examining this question, note that if the discrepancy $|x_A - x_B|$ between the two measurements is much greater than both uncertainties $\sigma_A$ and $\sigma_B$, we should suspect that something has gone wrong in at least one of the measurements. In this situation, we would say that the two measurements are inconsistent, and we should examine both measurements carefully to see whether either (or both) was subject to unnoticed systematic errors.

Let us suppose, however, that the two measurements (7.1) and (7.2) are consistent; that is, the discrepancy $|x_A - x_B|$ is not significantly larger than both $\sigma_A$ and $\sigma_B$. We can then sensibly ask what the best estimate $x_{\text{best}}$ is of the true value $X$, based on the two measurements. Your first impulse might be to use the average $(x_A + x_B)/2$ of the two measurements. Some reflection should suggest, however, that this average is unsuitable if the two uncertainties $\sigma_A$ and $\sigma_B$ are unequal. The simple
average \((x_a + x_b)/2\) gives equal importance to both measurements, whereas the more precise reading should somehow be given more weight.

Throughout this chapter, I will assume all systematic errors have been identified and reduced to a negligible level. Thus, all remaining errors are random, and the measurements of \(x\) are distributed normally around the true value \(X\).

### 7.2 The Weighted Average

We can solve our problem easily by using the principle of maximum likelihood, much as we did in Section 5.5. We are assuming that both measurements are governed by the Gauss distribution and denote the unknown true value of \(x\) by \(X\). Therefore, the probability of Student \(A\)‘s obtaining his particular value \(x_a\) is

\[
\text{Prob}_A(x_a) = \frac{1}{\sigma_a} e^{-\frac{(x_a - X)^2}{2\sigma_a^2}}, \tag{7.3}
\]

and that of \(B\)‘s getting his observed \(x_b\) is

\[
\text{Prob}_B(x_b) = \frac{1}{\sigma_b} e^{-\frac{(x_b - X)^2}{2\sigma_b^2}}. \tag{7.4}
\]

The subscript \(X\) indicates explicitly that these probabilities depend on the unknown actual value.

The probability that \(A\) finds the value \(x_a\) and \(B\) the value \(x_b\) is just the product of the two probabilities (7.3) and (7.4). In a way that should now be familiar, this product will involve an exponential function whose exponent is the sum of the two exponents in (7.3) and (7.4). We write this as

\[
\text{Prob}(x_a, x_b) = \text{Prob}_A(x_a) \text{Prob}_B(x_b) = \frac{1}{\sigma_a \sigma_b} e^{-\frac{(x_a - X)^2}{2\sigma_a^2} - \frac{(x_b - X)^2}{2\sigma_b^2}}, \tag{7.5}
\]

where I have introduced the convenient shorthand \(\chi^2\) (chi squared) for the exponent

\[
\chi^2 = \frac{(x_a - X)^2}{\sigma_a^2} + \frac{(x_b - X)^2}{\sigma_b^2}. \tag{7.6}
\]

This important quantity is the sum of the squares of the deviations from \(X\) of the two measurements, each divided by its corresponding uncertainty.

The principle of maximum likelihood asserts, just as before, that our best estimate for the unknown true value \(X\) is that value for which the actual observations \(x_a, x_b\) are most likely. That is, the best estimate for \(X\) is the value for which the probability (7.5) is maximum or, equivalently, the exponent \(\chi^2\) is minimum. (Because maximizing the probability entails minimizing the “sum of squares” \(\chi^2\); this method for estimating \(X\) is sometimes called the “method of least squares.”) Thus, to find the best estimate, we simply differentiate (7.6) with respect to \(X\) and set the derivative equal to zero,

\[
2 \frac{x_a - X}{\sigma_a^2} + 2 \frac{x_b - X}{\sigma_b^2} = 0.
\]
The solution of this equation for \( X \) is our best estimate and is easily seen to be

\[
(\text{best estimate for } X) = \left( \frac{x_A}{\sigma_A^2} + \frac{x_B}{\sigma_B^2} \right) \left/ \left( \frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2} \right) \right.
\]

(7.7)

This rather ugly result can be made tidier if we define weights

\[
w_A = \frac{1}{\sigma_A^2} \quad \text{and} \quad w_B = \frac{1}{\sigma_B^2}.
\]

(7.8)

With this notation, we can rewrite (7.7) as the weighted average (denoted \( x_{\text{av}} \))

\[
(\text{best estimate for } X) = x_{\text{av}} = \frac{w_A x_A + w_B x_B}{w_A + w_B}
\]

(7.9)

If the original two measurements are equally uncertain (\( \sigma_A = \sigma_B \) and hence \( w_A = w_B \)), this answer reduces to the simple average \((x_A + x_B)/2\). In general, when \( w_A \neq w_B \), the weighted average (7.9) is not the same as the ordinary average; it is similar to the formula for the center of gravity of two bodies, where \( w_A \) and \( w_B \) are the actual weights of the two bodies, and \( x_A \) and \( x_B \) their positions. In (7.9), the “weights” are the inverse squares of the uncertainties in the original measurements, as in (7.8). If \( A \)'s measurement is more precise than \( B \)'s, then \( \sigma_A < \sigma_B \) and hence \( w_A > w_B \), so the best estimate \( x_{\text{av}} \) is closer to \( x_A \) than to \( x_B \), just as it should be.

**Quick Check 7.1.** Workers from two laboratories report the lifetime of a certain particle as \( 10.0 \pm 0.5 \) and \( 12 \pm 1 \), both in nanoseconds. If they decide to combine the two results, what will be their respective weights as given by (7.8) and their weighted average as given by (7.9)?

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Our analysis of two measurements can be generalized to cover any number of measurements. Suppose we have \( N \) separate measurements of a quantity \( x \),

\[
x_1 \pm \sigma_1, \quad x_2 \pm \sigma_2, \ldots, x_N \pm \sigma_N,
\]

with their corresponding uncertainties \( \sigma_1, \sigma_2, \ldots, \sigma_N \). Arguing much as before, we find that the best estimate based on these measurements is the weighted average

\[
x_{\text{av}} = \frac{\sum w_i x_i}{\sum w_i}
\]

(7.10)

where the sums are over all \( N \) measurements, \( i = 1, \ldots, N \), and the weight \( w_i \) of each measurement is the reciprocal square of the corresponding uncertainty,

\[
w_i = \frac{1}{\sigma_i^2}
\]

(7.11)

for \( i = 1, 2, \ldots, N \).
Because the weight \( w_j = 1/\sigma_j^2 \) associated with each measurement involves the square of the corresponding uncertainty \( \sigma_j \), any measurement that is much less precise than the others contributes very much less to the final answer (7.10). For example, if one measurement is four times less precise than the rest, its weight is 16 times less than the other weights, and for many purposes this measurement could simply be ignored.

Because the weighted average \( x_{\text{avg}} \) is a function of the original measured values \( x_1, x_2, \ldots, x_n \), the uncertainty in \( x_{\text{avg}} \) can be calculated using error propagation. As you can easily check (Problem 7.8), the uncertainty in \( x_{\text{avg}} \) is

\[
\sigma_{x_{\text{avg}}} = \frac{1}{\sqrt{\Sigma w_i}} \tag{7.12}
\]

This rather ugly result is perhaps a little easier to remember if we rewrite (7.11) as

\[
\sigma_j = \frac{1}{\sqrt{w_j}} \tag{7.13}
\]

Paraphrasing Equation (7.13), we can say that the uncertainty in each measurement is the reciprocal square root of its weight. Returning to Equation (7.12), we can paraphrase it similarly to say that the uncertainty in the grand answer \( x_{\text{avg}} \) is the reciprocal square root of the sum of all the individual weights; in other words, the total weight of the final answer is the sum of the individual weights \( w_j \).

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Quick Check 7.2. What is the uncertainty in your final answer for Quick Check 7.1?

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### 7.3 An Example

Here is an example involving three separate measurements of the same resistance.

**Example: Three Measurements of a Resistance**

Each of three students measures the same resistance several times, and their three final answers are (all in ohms):

- Student 1: \( R = 11 \pm 1 \)
- Student 2: \( R = 12 \pm 1 \)
- Student 3: \( R = 10 \pm 3 \)

Given these three results, what is the best estimate for the resistance \( R \)?

The three uncertainties \( \sigma_1, \sigma_2, \sigma_3 \) are 1, 1, and 3. Therefore, the corresponding weights \( w_j = 1/\sigma_j^2 \) are

\[
w_1 = 1, \quad w_2 = 1, \quad w_3 = \frac{1}{3}.
\]
The best estimate for $R$ is the weighted average, which according to (7.10) is

$$R_{\text{avr}} = \frac{\sum w_i R_i}{\sum w_i} = \frac{(1 \times 11) + (1 \times 12) + (1 \times 10)}{1 + 1 + \frac{1}{4}} = 11.42 \text{ ohms.}$$

The uncertainty in this answer is given by (7.12) as

$$\sigma_{R_{\text{avr}}} = \frac{1}{\sqrt{\sum w_i}} = \frac{1}{\sqrt{1 + 1 + \frac{1}{4}}} = 0.69.$$

Thus, our final conclusion (properly rounded) is

$$R = 11.4 \pm 0.7 \text{ ohms.}$$

For interest, let us see what answer we would get if we were to ignore completely the third student’s measurement, which is three times less accurate and hence nine times less important. Here, a simple calculation gives $R_{\text{avr}} = 11.50$ (compared with 11.42) with an uncertainty of 0.71 (compared with 0.69). Obviously, the third measurement does not have a big effect.

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### Principal Definitions and Equations of Chapter 7

If $x_1, x_2, \ldots, x_N$ are measurements of a single quantity $x$, with known uncertainties $\sigma_1, \sigma_2, \ldots, \sigma_N$, then the best estimate for the true value of $x$ is the weighted average

$$x_{\text{avr}} = \frac{\sum w_i x_i}{\sum w_i}, \quad [\text{See (7.10)}]$$

where the sums are over all $N$ measurements, $i = 1, \ldots, N$, and the weights $w_i$ are the reciprocal squares of the corresponding uncertainties,

$$w_i = \frac{1}{\sigma_i^2}.$$

The uncertainty in $x_{\text{avr}}$ is

$$\sigma_{x_{\text{avr}}} = \frac{1}{\sqrt{\sum w_i}}, \quad [\text{See (7.12)}]$$

where, again, the sum runs over all of the measurements $i = 1, 2, \ldots, N$.