Appendix D

Probabilities for Chi Squared

If a series of measurements is grouped into bins $k = 1, \ldots, n$, we denote by O_k the number of measurements observed in the bin k. The number *expected* (on the basis of some assumed or expected distribution) in the bin k is denoted by E_k. The extent to which the observations fit the assumed distribution is indicated by the reduced chi squared, $\tilde{\chi}^2$, defined as

$$\tilde{\chi}^2 = \frac{1}{d} \sum_{k=1}^{n} \frac{(O_k - E_k)^2}{E_k},$$

where d is the number of degrees of freedom, $d = n - c$, and c is the number of constraints (see Section 12.3). The expected average value of $\tilde{\chi}^2$ is 1. If $\tilde{\chi}^2 \gg 1$, the observed results do not fit the assumed distribution; if $\tilde{\chi}^2 \lesssim 1$, the agreement is satisfactory.

This test is made quantitative with the probabilities shown in Table D. Let $\tilde{\chi}_o^2$ denote the value of $\tilde{\chi}^2$ actually obtained in an experiment with d degrees of freedom. The number $Prob_d(\tilde{\chi}^2 \geqslant \tilde{\chi}_o^2)$ is the probability of obtaining a value of $\tilde{\chi}^2$ as large as the observed $\tilde{\chi}_o^2$, if the measurements really did follow the assumed distribution. Thus, if $Prob_d(\tilde{\chi}^2 \geqslant \tilde{\chi}_o^2)$ is large, the observed and expected distributions are consistent; if it is small, they probably disagree. In particular, if $Prob_d(\tilde{\chi}^2 \geqslant \tilde{\chi}_o^2)$ is less than 5%, we say the disagreement is *significant* and reject the assumed distribution at the 5% level. If it is less than 1%, the disagreement is called *highly significant,* and we reject the assumed distribution at the 1% level.

For example, suppose we obtain a reduced chi squared of 2.6 (that is, $\tilde{\chi}_o^2 = 2.6$) in an experiment with six degrees of freedom ($d = 6$). According to Table D, the probability of getting $\tilde{\chi}^2 \geqslant 2.6$ is 1.6%, if the measurements were governed by the assumed distribution. Thus, at the 5% level (but not quite at the 1% level), we would reject the assumed distribution. For further discussion, see Chapter 12.

Table D. The percentage probability $Prob_d(\tilde{\chi}^2 \geqslant \tilde{\chi}_o^2)$ of obtaining a value of $\tilde{\chi}^2 \geqslant \tilde{\chi}_o^2$ in an experiment with d degrees of freedom, as a function of d and $\tilde{\chi}_o^2$. (Blanks indicate probabilities less than 0.05%.)

	$\tilde{\chi}_o^2$														
d	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	8.0	10.0
1	100	48	32	22	16	11	8.3	6.1	4.6	3.4	2.5	1.9	1.4	0.5	0.2
2	100	61	37	22	14	8.2	5.0	3.0	1.8	1.1	0.7	0.4	0.2		
3	100	68	39	21	11	5.8	2.9	1.5	0.7	0.4	0.2	0.1			
4	100	74	41	20	9.2	4.0	1.7	0.7	0.3	0.1	0.1				
5	100	78	42	19	7.5	2.9	1.0	0.4	0.1						

d	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
1	100	65	53	44	37	32	27	24	21	18	16	14	12	11	9.4	8.3
2	100	82	67	55	45	37	30	25	20	17	14	11	9.1	7.4	6.1	5.0
3	100	90	75	61	49	39	31	24	19	14	11	8.6	6.6	5.0	3.8	2.9
4	100	94	81	66	52	41	31	23	17	13	9.2	6.6	4.8	3.4	2.4	1.7
5	100	96	85	70	55	42	31	22	16	11	7.5	5.1	3.5	2.3	1.6	1.0
6	100	98	88	73	57	42	30	21	14	9.5	6.2	4.0	2.5	1.6	1.0	0.6
7	100	99	90	76	59	43	30	20	13	8.2	5.1	3.1	1.9	1.1	0.7	0.4
8	100	99	92	78	60	43	29	19	12	7.2	4.2	2.4	1.4	0.8	0.4	0.2
9	100	99	94	80	62	44	29	18	11	6.3	3.5	1.9	1.0	0.5	0.3	0.1
10	100	100	95	82	63	44	29	17	10	5.5	2.9	1.5	0.8	0.4	0.2	0.1
11	100	100	96	83	64	44	28	16	9.1	4.8	2.4	1.2	0.6	0.3	0.1	0.1
12	100	100	96	84	65	45	28	16	8.4	4.2	2.0	0.9	0.4	0.2	0.1	
13	100	100	97	86	66	45	27	15	7.7	3.7	1.7	0.7	0.3	0.1	0.1	
14	100	100	98	87	67	45	27	14	7.1	3.3	1.4	0.6	0.2	0.1		
15	100	100	98	88	68	45	26	14	6.5	2.9	1.2	0.5	0.2	0.1		
16	100	100	98	89	69	45	26	13	6.0	2.5	1.0	0.4	0.1			
17	100	100	99	90	70	45	25	12	5.5	2.2	0.8	0.3	0.1			
18	100	100	99	90	70	46	25	12	5.1	2.0	0.7	0.2	0.1			
19	100	100	99	91	71	46	25	11	4.7	1.7	0.6	0.2	0.1			
20	100	100	99	92	72	46	24	11	4.3	1.5	0.5	0.1				
22	100	100	99	93	73	46	23	10	3.7	1.2	0.4	0.1				
24	100	100	100	94	74	46	23	9.2	3.2	0.9	0.3	0.1				
26	100	100	100	95	75	46	22	8.5	2.7	0.7	0.2					
28	100	100	100	95	76	46	21	7.8	2.3	0.6	0.1					
30	100	100	100	96	77	47	21	7.2	2.0	0.5	0.1					

The values in Table D were calculated from the integral

$$Prob_d(\tilde{\chi}^2 \geqslant \tilde{\chi}_o^2) = \frac{2}{2^{d/2}\Gamma(d/2)} \int_{\chi_o}^{\infty} x^{d-1} e^{-x^2/2} dx.$$

See, for example, E. M. Pugh and G. H. Winslow, *The Analysis of Physical Measurements* (Addison-Wesley, 1966), Section 12-5.